know that we have supplanted the compasses by a far simplei instrument, the sect-carrier, and that again by the unitsect-carrier.

GEORGE BRUCE HALSTED.

GREELEY, COL.

CONSTRUCTION OF THE STRAIGHT LINE.

IN COMMENT ON MR. FRANCIS C. RUSSELL'S ARTICLE[1] "A MODERN ZENO."

Mathematicians will take an interest in Francis C. Russell's attack on the mathematical system of Lobatchevsky, whom he calls a "modern Zeno." If Mr. Russell is right we shall have to grant that there is a flaw in the arguments of Lobatchevsky on which he bases a new geometry that in contrast to Euclid's does not acknowledge the postulate of parallel lines.

Mr. Chas. S. Peirce in a letter to Mr. Russell thinks that he (Mr. Russell) overshot the mark. He says: "Those two lines cutting each other are *not* parallel and his (Lobatchevsky's) defining them as parallel to the third was in obvious contradiction to the proposition that two straight lines both parallel to a third are necessarily parallel to each other. I press the question, Why did you not content yourself with this obvious proof of the incorrectness of his proposition No. 25? The answer seems to me obvious. If you had done that your readers would have at once perceived that Lobatchevsky merely made a slip of the pen and *meant* that two straight lines parallel to a third *toward the same side* are parallel to each other."

Though Mr. Russell may have gone too far, he has called attention to a mistake which ought to be corrected, and Mr. Charles S. Peirce, in thoughtful consideration of the difficulty which puzzled Mr. Russell, points out the flaw.

But metageometricians are not so considerate. They claim that he has thoroughly misunderstood non-Euclidean geometry. We publish in the present number two criticisms, one by Professor G. B. Halsted, the other by W. H. Bussey, assistant professor of mathematics at the University of Minnesota.

Metageometricians are a hotheaded race and display sometimes all the characteristics of sectarian fanatics. To them it is quite clear that there may be two straight lines through one and the same point which do not coincide and yet are both parallel to a third

[1] See the April number of *The Monist*.

straight line. I do not mean to take issue here for either Euclideans or non-Euclideans but I wish to say that the subject is difficult, that mathematicians are by no means so positively agreed on the subject as some metageometricians claim. If Mr. Russell is wrong, the admirers of Lobatchevsky are welcome to point out the mistakes in his objections. Mr. Russell has made no positive assertions, he has expressed his incredulity as to the soundness of Lobatchevsky's arguments and asks for further information on the subject. The problems of non-Euclidean geometry are not quite so simple, nor the solutions of Lobatchevsky so self-evident that a modest question on the subject would not be in order; but the editor is seriously requested to submit manuscripts to a mathematician (presumably an orthodox non-Euclidean) and to suppress all heretical articles. In reply to this request I will state that I frequently publish articles setting forth views which I do not endorse, because I believe that they are worth being noticed, considered and perhaps refuted. Mr. Russell, for instance, raises another issue (viz., the problem of a construction of the straight line) on which the greatest mathematicians have made the most divergent statements.

Leaving the discussion of Lobatchevsky's geometry to the non-Euclideans I wish now to criticise Mr. Russell for his construction of the straight line.

Mr. Russell attempts to define and develop the straight line by purely *a priori* methods and does it without the ruler, limiting his method to the use of the compasses. He constructs three spheres, and by the use of the compasses only he lays down a range of points which in their totality mark a straight line. Incidentally he refers appreciatively to my book on the *Foundations of Mathematics,* and I gladly note many points of agreement which, however, Mr. Russell has worked out in perfect independence. Like myself Mr. Russell calls attention to the significance of even-boundary conceptions the value of which consists in their uniqueness, and he is pleased with the term "anyness"; but I would suggest that if he had adopted my view of the foundation of mathematics, he would have deemed it redundant to construct the straight line as he does, and would be satisfied to produce it (as I have done) as an even-boundary conception; for after all he shares the mistake of all attempts of the same kind, in that while constructing the straight line, he presupposes it. He says most impressively when speaking of the indispensableness of the straight line (and I subscribe to every word of it): "All things in mathematics have been made by it and without

it has not been made anything that has been made." But even while making the statement Mr. Russell forgets this truth for a moment and inadvertently proves it in his very construction of the straight line, for he presupposes and uses conditions which involve the straight line, while he attempts to lay it down with the help of the compasses.

The same idea, at least in its principle, has been suggested before by Fourier who proposed a new construction of the straight line in the following way. We quote from an article by G. B. Halsted in *The Monist,* IV, p. 485:

"Take any two points on any solid. Let one remain at rest while the solid moves. The other describes a sphere. Two spheres intersect in a circle. If the spheres are equal and grow, this circle describes a plane. If the spheres touch and one decreases as the other grows, their point of contact describes a straight."

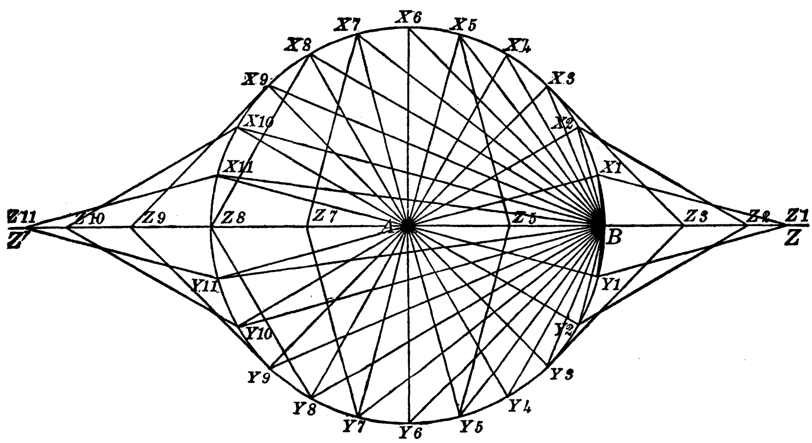
Fourier's construction of the straight line suffers from the same faults as that of Mr. Russell. Both presuppose the straight line, both are constructed in a homaloidal space, under conditions of anyness, which renders the distance between two points definite. This definite distance between two points is determinable (i. e., measurable) only by a straight line. If we could not measure distance so as to be sure that it does not change while the moving point travels around the stationary point, there would be no use of the construction.

Almost every metageometrician remains unaware that everything he does he accomplishes through the instrumentality of the straight line, and that the straight line is indispensable even if we draw a circle. Here we have good evidence of Mr. Russell's dictum concerning the straight line, that "all things in mathematics have been made by it and without it has not been made anything that has been made."

Mr. Russell, as well as M. Fourier, starts with the construction of a sphere and naturally makes use of the radius. But what is the radius but a straight line, the straight line being the measure of the distance between two points? When we lay down two points at a definite distance we imply the straight line which is our only means of uniquely, i. e., unequivocally, determining distance, otherwise we have no means to distinguish radii of different lengths. It is evident that these two constructions, Mr. Russell's and M. Fourier's as well as all others which produce the straight line by some such legerdemain, presuppose the notion of an even space, or of distance that remains

the same, or of a scope of motion under conditions of anyness. All three being different expressions for practically the same thing.

The issue which I raise is no quibbling and will be driven home to the reader who would try to construct the straight line with a pair of compasses that are not firmly set. He will have again and again to assure himself that the distance has remained the same. When we construct circles we presuppose an even (or homaloidal) scope of motion. We presuppose that distances are definite and measurable. We presuppose the existence and workableness of the compasses. The ruler is first and the compasses second. The circle, being begotten of the radius, presupposes the straight line. In fact the compasses determine the size of a straight line, for the essential part of the compasses consists in the adjustability of its two points,



not in the two legs. The two legs are merely a convenience. They are the machinery to fix the points and a handle to turn them in their fixed position. We might as well use a string pinned down at one end and having a pencil at the other; and what is a string stretched tight if not a materialization of the straight line?

We here reproduce Mr. Russell's diagram which shows on two circles what he proposes to do with three spheres for the sake of developing the straight line by means of the compasses only and without the ruler. In order to show the several openings of the compasses used, he draws the radii and thus makes visible what they involve. Just look at all these straight lines which are here introduced as auxiliary constructions, and there are still more of them doing obstetrical service for the birth of the straight line from the

cooperation of the three spheres. The very spheres themselves have been begotten by the straight line, which first performing the function of a radius, made one end stay in one place (the center) and let the other swing around it; then having created the circle it was again the straight line which as a diameter of the circle served as an axis of its rotation so as to produce the sphere. Verily Mr. Russell is right and we repeat his proposition with religious solemnity. All things in mathematics have been made by the straight line.

Mr. Russell's contention would be proved only if he could make his construction with the circle alone and dispense with the ruler entirely; he should also dispense with it in his proof. But he can not. His construction does not create a straight line; in fact it creates no line at all, but only (as he says himself) a range of points, and all we have to grant is that his range of points lies in a straight line. But how does he prove it? How do we know and in what way can the site of this range of points as being in a straight line, be determined? We can determine it only by having a clear conception of a straight line and bringing it to bear on our range of points. We must make the straight line run through the range of points thus constructed by Mr. Russell and prove that they all lie in the path of the straight line. In other words, any range of points does not constitute a line, and unless we have the idea of a straight line, we can not bridge the distance between any two points (let alone a great number of points) and then declare that we have accomplished the task.

The fundamental error of Mr. Russell, M. Fourier, and all who have made kindred attempts, consists in the assumption that mathematics has to start from a blank and is an *a priori* construction out of nothing. Mathematics starts from an absence of all concrete existence, and this can be called "nothing" only in a certain sense. The domain of mathematics is a nothingness in the sense of an absence of all materiality, of all forces, of energy, of all bodily existences, and of all concreteness. As I have expressed it in my *Foundations of Mathematics,* the mathematician starts from a state of "anyness" and this absence of all concrete existence is not an absolute nothing. Anyness involves homogeneity and homogeneity is the characteristic feature of mathematical space—the scope of motion for the mathematician's operations.

The mathematician performs operations, but his operations are pure motions of anyness, which means they are stripped of all par-

ticularity and concreteness. They are devoided of matter and energy with all their qualities. Thus the determination of a locus is a mere point without extension and its motion produces mere length without breadth or thickness, etc. Everywhere we meet with that subtle fabric of anyness which is a true nothing in the sense of the absence of everything concrete, but not an absolute nothing. In this anyness the mathematician operates and his mode of operation is a work of anyness.

Mathematical space which is the domain of anyness in which the mathematician performs his operations, includes the possibility of constructing even boundaries, and even boundaries are needed for mathematical constructions on account of their quality of being unique. Uniqueness is needed in order to have a standard of reference. The three even boundaries which thus recommend themselves by their uniqueness as standards of reference, are the straight line, the plane, and the right angle, and they make it possible to construct parallel lines. Accordingly it is obvious that the problems of the straight line, of the plane, of the right angle, of the sum of the angles in a triangle as equal to two right angles, and of parallelism are practically the same problem, and it is impossible to construct any one of them from nothing with the help of pure logic only. In addition to pure logic, the mathematician needs for the construction of his science the concept of anyness which yields that most indispensable quality of mathematical space, homogeneity without which mathematics would be impossible.

This idea of anyness is a product of abstraction and the mathematician should know its origin as well as its application in order to understand the foundation of his science.

EDITOR.

SOME REMARKS ON MR. RUSSELL'S ARTICLE, "A MODERN ZENO."

I have been reading with interest the April number of *The Monist,* especially "The Choice of Facts," by H. Poincaré, and "A Newly Discovered Treatise of Archimedes," by J. L. Heiberg. I was attracted by the title "A Modern Zeno," and I was very much surprised to learn the identity of the man. Mr. Russell, the writer of the article, has evidently made some study of Non-Euclidean Geometry, especially of the writings of Lobatchevsky. But truly "a little learning is a dangerous thing." His study has been super-